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THE RELATION BETWEEN THE MECHANICS OF DISSIPATIVE FINITE-DIMENSIONAL SYSTEMS WITH HEREDITY AND THE MECHANICS OF INFINITE-DIMENSIONAL HAMILTONIAN SYSTEMS†

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It is shown that the dynamics of a non-linear multi-dimensional oscillator interacting with a field of harmonic oscillators, continuously distributed with respect to frequency, is governed by a non-linear integro-differential equation. The investigation centres on the possibility of an inverse transition from a non-linear oscillator with heredity to be embeddable in a larger Hamiltonian system is the usual condition that the entropy production should not be negative. Existence and uniqueness theorems are proved and several a priori estimates are found for the solution. It is also proved that, subject to certain restrictions on the relaxation kernel, the solution converges to one of the critical points of the effective potential. © 1999 Elsevier Science Ltd. All rights reserved.

It is usually assumed that the description of dissipative phenomena (such as friction, viscosity, relaxation etc.) lies beyond the scope of Hamiltonian mechanics. Indeed, Poincaré's recurrence theorem [1] might seem to exclude the possibility of irreversible processes for Hamilton's equations. However, the validity of Poincaré's theorem is related in an essential way to the condition that the number of degrees of freedom be finite. If the system is infinite-dimensional, a trajectory in a general position no longer has to return to an arbitrarily small neighbourhood of the initial state.

The idea of describing dissipative processes within the framework of Hamiltonian mechanics is therefore as follows. We assume that the set of degrees of freedom of the system may be divided into two subsets, so that the phase space may be represented as a produce $S_1 \times S_2$, where S_1 is a finite-dimensional set and S_2 is infinite-dimensional. The initial conditions for the degrees of freedom in S_2 are specified. Solving the Hamilton's equations, one can eliminate the characteristics of the subsystem corresponding to S_2, and obtain a single dynamical system of equations for the subsystem corresponding to S_1. This last system of equations is non-local in time, that is, it involves heredity effects and may describe dissipative phenomena. This technique is well known for models with quadratic Hamiltonians which admit of exact solutions [2–4], that is, for linear equations. In what follows we will investigate whether dissipative processes can be described in Hamiltonian mechanics for the non-linear case.

1. THE RELATIONSHIP BETWEEN LAGRANGIAN SYSTEMS AND SYSTEMS WITH HEREDITY

We will use the usual definition of a scalar product and norm in $\mathbb{C}^N$

$$(z,z') = \sum_{i=1}^{N} z_i^* z_i', \quad |z| = (z,z)^{1/2}$$

$$z = (z_i), \quad z' = (z_i'), \quad i = 1, \ldots, N$$

The asterisk in the superscript position denotes the operation of complex conjugation, A^+ is the adjoint matrix to A and f^*g is the convolution with respect to time of the time functions $f = f(t)$ and $g = g(t)$. For real matrices, of course, the conjugation operation is identical with ordinary transposition. The norm $|\cdot|$ in ^{N} induces a norm $\|\cdot\|$, in the usual manner, for linear operators in $\mathbb{C}^N$.

Consider a Lagrangian system with coordinates

$$x = (x_i) = (x_i(t)) \in \mathbb{R}^N, \quad \varphi = (\varphi_i) = (\varphi_i(t,\omega)) \in \mathbb{R}^N$$

$$i = 1, \ldots, N, \quad \omega \in [0, +\infty)$$

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and Lagrangian

$$L=\frac{1}{2}(\partial_t x,\partial_t x)-V(x)+\frac{1}{2}\int_0^\infty((\partial_t\varphi,\partial_t\varphi)-\omega^2(\varphi,\varphi))d\omega+$$

$$+\int_0^\infty (x,\ q\varphi)d\omega+(f(t),\ x) \tag{1.1}$$

The physical meaning of this system is that a selected N-dimensional non-linear oscillator $x = x(t)$ is interacting with a field of N-dimensional harmonic oscillators $\varphi = \varphi(t, \omega)$, which are continuously distributed with respect to the frequency ω. Here $V = V(x)$ is the potential energy of the oscillator, $f = ((f_i(t))$ is the external force and $q = q(\omega)$ is a real weighting matrix, not identically zero, characterizing the interaction.

Throughout, we will assume that: (1) $V\in C^2(\mathbb{R}^N)$, (2) $q \in C^1[0, +\infty)$, (3) the matrix-valued function $dq/d\omega = dq(\omega)/d\omega$ has a right derivative at $\omega = 0$ and 4) $\int_0^\infty\omega^{-1}\,\mathrm{Tr}(qq^+)d\omega < +\infty$. The last condition implies that $q(0) = 0$. It is convenient to extend the definition of the matrix-valued function $q = q(\omega)$ to negative values of the argument: $q(\omega) = q(-\omega)$, $\omega < 0$.

We will denote the vector of first derivatives of the potential $V = V(x)$ by ∇V and the matrix of second derivatives of V by $(\dot{\nabla}\nabla V)$.

If the external forces are zero, the potential energy of the system is given by the expression

$$U=V+\frac{1}{2}\int_0^\infty\omega^2(\varphi,\ \varphi)d\omega-\int_0^\infty(x,\ q\varphi)d\omega=V_1+\frac{1}{2}\int_0^\infty\omega^2(\varphi-\omega^{-2}q^+x,\ \varphi-\omega^{-2}q^+x)d\omega$$

$$V_1=V_1(x)=V(x)-\frac{1}{2}(x,\ \gamma x),\quad \gamma=\int_0^\infty\omega^{-2}qq^+d\omega$$

Consequently, a necessary and sufficient condition for the system to be energetically stable is that the function $V_1 = V_1(x)$ should have a lower bound. In what follows we will assume a stronger condition

$$V_1(x)\to+\infty \quad \text{as } |x|\to+\infty \tag{1.2}$$

The Lagrange equations follow from (1.1)

$$\partial_t^2x+\nabla V=\int_0^{+\infty}q\varphi d\omega+f,\quad \partial_t^2\varphi+\omega^2\varphi=q^+x \tag{1.3}$$

We will seek a solution of system (1.3) when $t \geq 0$ satisfying the initial conditions

$$x(0)=x_0,\quad \partial_t x(0)=y_0 \tag{1.4}$$

$$\varphi(0,\ \omega)=\varphi_0(\omega),\quad \partial_t\varphi(0,\ \omega)=\psi_0(\omega) \tag{1.5}$$

The second equation of (1.4) and conditions (1.5) imply the following expression for the oscillators of the field

$$\varphi(t,\ \omega)=\omega^{-1}q(\omega)^+\int_0^t\sin\omega(t-t_1)x(t_1)dt_1+\chi(t,\ \omega) \tag{1.6}$$

$$\chi(t,\ \omega)=\varphi_0(\omega)\cos\omega t+\psi_0(\omega)\omega^{-1}\sin\omega t$$

Substituting this expression into the first equation of (1.3), we obtain an integro-differential equation

$$\partial_t^2x(t)+\nabla V_1(x(t))+\int_0^t K(t-t_1)\partial_t x(t_1)dt_1=f_1(t) \tag{1.7}$$

where

$$K(t)=\int_0^\infty\omega^{-2}q(\omega)q(\omega)^+\cos\omega t d\omega,\quad t\geq 0 \tag{1.8}$$

$$f_1(t) = f(t) - K(t)x_0 + \int_0^\infty q(\omega)\chi(t,\ \omega)d\omega \tag{1.9}$$

It is convenient to complete the definition of the kernel for negative times: $K(t) = 0, t < 0$.

Note that for slow processes Eq. (1.7) becomes the equation of forced oscillations of a multi-dimensional non-linear oscillator with friction

$$\partial_t^2 x + \nabla V_1 + \lambda\partial_t x = f_1(t), \quad \lambda = \int_0^\infty K(t)dt$$

Thus Eq. (1.7) is a generalization of the usual equation for a damped oscillator. We will show later that the solutions of Eq. (1.7) have many properties typical of ordinary damped oscillations.

We recall that if $g = g(t)$ is a generalized function (or distribution) of moderate growth, then its Fourier transform is defined [5]; we will denote it by

$$g_F(\Omega) = \int e^{-i\Omega t} g(t)dt$$

Consequently, the Fourier transform $K_F(\Omega)$ of the kernel is defined. By the Paley–Wiener theorem, it is analytic in the lower complex half-plane. Using (1.8), we find that

$$K_F(\Omega) = \frac{1}{2i}\int_{-\infty}^{+\infty} \omega^{-2} q(\omega)q(\omega)^+(\Omega-\omega)^{-1}d\omega, \quad \operatorname{Im}\Omega < 0 \tag{1.10}$$

Given the usual assumptions concerning the function $q = q(\omega)$, when taking limits with respect to real values of Ω in this expression, we can apply the Sokhotskii–Plemelj formula [6]

$$K_F(\Omega) = \frac{1}{2i}\mathrm{V.p.}\int_{-\infty}^{+\infty} \omega^{-2} q(\omega)q(\omega)^+(\Omega-\omega)^{-1}d\omega + \frac{\pi}{2}\Omega^{-2}q(\Omega)q^+(\Omega) \tag{1.11}$$

At the point $\Omega = 0$ the right-hand side of (1.11) is defined by continuity.

We have shown that, by reduction with respect to some of the degrees of freedom, the Lagrangian system (1.1) may be reduced to the form of a non-linear oscillator with heredity (1.7). We will now consider the converse process: given an oscillator with heredity (1.7), we will try to embed it in some Lagrangian system with a large set of degrees of freedom.

Lemma 1. Let $K = K(t) = K(t)^+$ be a real matrix-valued function, continuous on the half-line $t \geqslant 0$ and vanishing at $t < 0$. Suppose moreover that the integrals

$$k = \int_0^{+\infty} \| K(t) \| dt < +\infty, \qquad \int_{-\infty}^{+\infty} \| K_F(\Omega) \| d\Omega < +\infty \tag{1.12}$$

are convergent and that for real values of Ω

$$K_F(\Omega) + K_F(\Omega)^+ \geqslant 0 \tag{1.13}$$

Assume, furthermore, that in some real neighbourhood of $\Omega = 0$

$$K_F(\Omega) = a_0 + a_1 |\Omega| + o(\Omega)$$

for some real symmetric matrices a_0 and a_1.

Then a function $q = q(\omega)$, exists connected with the function $K = K(t)$ by formula (1.11), and satisfying all the assumptions adopted for this function.

Proof. Following (1.11), we will seek a function $q = q(\omega)$ satisfying the equation

$$q(\Omega)q^+(\Omega) = \pi^{-1}\Omega^{-2}(K_F(\Omega) + K_F(\Omega)^+)$$

It follows from the assumptions of the lemma that this equation defines a (not necessarily unique) function $q = q(\omega)$ which satisfies all the required conditions. Equation (1.11) follows from the Sokhotskii–Plemelj formula. This completes the proof.

Remark 1. The main condition (1.13) for an oscillator with heredity to be embeddable in a broader Hamiltonian system corresponds to the requirement that the second law of thermodynamics should hold for systems with delay. Indeed, inequality (1.13) is equivalent to the inequality

$$\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 (y(t_1), \quad K(t_1 - t_2) y(t_2)) \geqslant 0$$

for arbitrary rapidly decreasing real vector-valued functions $y = y(t)$. This inequality corresponds precisely to the condition that the entropy production be non-negative [7, 8] in the mechanics of materials with a memory. The consistency of the model with the second law of thermodynamics automatically implies that it is embeddable in some broader Hamiltonian system.

We will continue with a few examples of the application of Eq. (1.7) in mechanics.

Example 1. *Small oscillations of a massive slab on a layer of compressible visco-elastic liquid.* Suppose a weightless layer of visco-elastic compressible liquid lies between a fixed base and a flat slab of mass M. The liquid is assumed to satisfy rheological relations in the form

$$p_{ij} = -p\delta_{ij} + \left(K_V - \frac{2}{3}K_S\right) * e_{kk}\delta_{ij} + 2K_S * e_{ij} \tag{1.14}$$

where e_{ij} is the tensor of deformation rates, $p = p(\rho)$ is the hydrostatic pressure as a function of the density ρ, and $K_V = K_V(t)$ and $K_S = K_S(t)$ are the relaxation kernels for bulk and shear deformations, respectively. Let $x = x(t)$ be the variable thickness of the layer, let x_0 and ρ_0 be the thickness and density of the layer at time zero, and let $f = f(t)$ be an external force applied to the slab (over the whole of the slab). The dynamics of the system are described by the equation

$$M\partial_t^2 x = p(x_0\rho_0 x^{-1}) - \left(K_V + \frac{4}{3}K_S\right) * x^{-1}\partial_t x + f(t)$$

If $|x - x_0|/x_0 \ll 1$, one can linearize the relaxation term and obtain an equation of the form (1.7). When this is done, the pressure may depend non-linearly on the layer thickness. The conditions Re $K_{VF} \geqslant 0$, Re $K_{SF} \geqslant 0$ are the usual dissipativeness conditions for visco-elastic materials [9].

Example 2. *The angular vibration of a shaft.* Suppose a massive shaft of radius R_1 can rotate about its own axis; let $\varphi = \varphi(t)$ denote the angle of rotation. Assume that the shaft is in contact with a visco-elastic lubricant filling the region $R_1 \leqslant r \leqslant R_2 (R_1 < R_2)$, and that the outer boundary of the flow region $r = R_2$ is fixed. Assuming that Couette flow occurs for the liquid with rheology (1.14), we obtain the equation

$$J\partial_t^2\varphi = -2\pi R_1^3 L(R_2 - R_1)^{-1} K_S * \partial_t\varphi + F(\varphi) + f(t) \tag{1.15}$$

where J is the moment of inertia of the shaft, $f(t)$ is the torque of the external forces, $F(\varphi)$ is a rotating torque set up by external devices and L is the shaft length. Equation (1.15) describes, for example, the operation of a rotation viscometer.

Example 3. *The oscillations of a particle in a medium with a microstructure.* In a medium with a microstructure, long waves may generate rotational vibration of micro-particles [10]

$$J\partial_t^2\varphi + \nabla V(\varphi) + \lambda\partial_t\varphi = f(t) \tag{1.16}$$

where $\varphi = (\varphi_i)$ is the micro-rotation vector, J is the density of the moment of inertia, $V = V(\varphi)$ is the elastic potential, λ is the coefficient of rotational viscosity and $f(t)$ is an external torque set up by a macroscopic elastic wave. To take the effects of heredity into account, the coefficient λ in Eq. (1.16) must be replaced by a convolution operator with a kernel [11]

$$J\partial_t^2\varphi + \nabla V(\varphi) + K * \partial_t\varphi = f(t)$$

A dissipativeness condition of type (1.13) is rigorously justified for this case in [11].

Throughout the remainder of this paper we will assume that the conditions of Lemma 1 are satisfied. In addition, we will replace condition (1.13) by the stronger inequality

$$K_F(\Omega) + K_F(\Omega)^+ \geqslant \rho(\Omega)\,\mathrm{id}_{\mathbb{R}^N} \tag{1.17}$$

where $\rho(\Omega)$ is a continuous positive function.

2. EXISTENCE AND UNIQUENESS THEOREMS AND A PRIORI ESTIMATES

We now prove a local existence and uniqueness theorem for problem (1.4), (1.7). To do this, we will use a method which is a simple extension of the Picard–Lindelöf method in the theory of ordinary differential equations [12].

Put

$$y = y(t) = \partial_t x(t), \quad z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\| z \| = \max(| x |, \ | y |), \quad \Phi(z) = \begin{pmatrix} y \\ -\nabla v_1(x) \end{pmatrix}$$

$$\Psi(t) = \begin{pmatrix} 0 & 0 \\ 0 & -K(t) \end{pmatrix}, \quad \varkappa(t) = \begin{pmatrix} 0 \\ f_1(t) \end{pmatrix}$$

Then problem (1.4), (1.7) may be rewritten as a first-order integro-differential equation

$$\partial_t z = w, \quad z(0) = z_0$$

$$w(t) = \Phi(z(t)) + \int_0^t \Psi(t - t_1) z(t_1) dt_1 + \varkappa(t)$$

or an integral equation

$$z(t) = z_0 + \int_0^t w(t_1) dt_1 \tag{2.1}$$

Theorem 1. Let the function $f_1 = f_1(t)$ be Lebesgue-integrable and bounded in the interval $[0, T_0]$. For any positive number a, we define

$$b_1 = \max_{|x - x_0| \leqslant a} (| \nabla V_1(x) |), \quad b = (b_0 + b_1 + (\| z_0 \| + a)k)$$

$b_0 = \sup_{0 < t < T_0} (|f_1(t)|)$ (the quantity k is defined by the first relation in (1.12)).

Then a unique solution of problem (2.1) exists, in the class of continuous functions, in the interval $[0, T_1]$, where $T_1 = \min(T_0, ab^{-1})$.

Proof. Existence. We will construct a solution for $0 \leqslant t \leqslant T_1$ by successive approximations

$$z_0(t) = z_0, \quad z_n(t) = z_0 + \int_0^t w_n(t_1) dt_1, \quad n > 0 \tag{2.2}$$

$$w_n(t) = \Phi(z_{n-1}(t)) + \int_0^t \Psi(t - t_1) z_{n-1}(t_1) dt_1 + \varkappa(t)$$

It can be proved by induction that

$$\| z_n(t) - z_0 \| \leqslant a \tag{2.3}$$

Indeed, suppose this inequality is true for $n = k \geqslant 0$.

Direct estimates then yield the inequality $\| w_{k+1}(t) \| \leqslant b$; using this inequality and the fact that $t \leqslant ab^{-1}$ we deduce (2.3) for $n = k + 1$ from (2.2).

Now let

$$c = \max_{|x - x_0| \leqslant a} (\| \nabla\nabla V_1(x) \|)$$

It can be proved by induction that

$$\| z_n(t) - z_{n-1}(t) \| \leqslant b(c + k)^{n-1} t^n / n! \tag{2.4}$$

Therefore, the sequence

$$z_n(t) = z_0 + \sum_{k=1}^{n} (z_k - z_{k-1})$$

converges uniformly to some continuous function $z = z(t)$. Hence, by virtue of our assumptions, the sequence of functions $w_n = w_n(t)$ converges uniformly to the function

$$\Phi(z(t)) + \int_0^t \Psi(t - t_1)z(t_1)dt_1 + \varkappa(t)$$

Hence, by the definition of the sequence $z_n(t)$, it follows that $z = z(t)$ is a solution of problem (2.1).

Uniqueness. Let $z_* = z_*(t)$ be some solution of problem (2.1) in the interval $[0, T_1]$. One proves by induction the inequality obtained from (2.4) when $z_{n-1}(t)$ is replaced by $z_*(t)$.
Hence it follows that $z(t) = z_*(t)$.

Remark 2. In what follows, a solution of problem (1.4), (1.7) will always be understood in the sense of Theorem 1. Generally speaking, Theorem 1 guarantees the local existence and uniqueness of a C^1 solution of problem (1.4), (1.7). However, if the function $f_1 = f_1(t)$ is piecewise continuous, then the function $y = y(t) = \partial_t x(t)$ is piecewise differentiable.

We will now consider the existence and uniqueness of a solution in the entire half-line $t \geqslant 0$. The usual way to obtain a global solution is by successive continuation of the solution using the local existence theorem. This method, however, can only be used when one has a priori bounds on the solution under construction. We will find such bounds.

Note that when there are no external forces the Lagrangian system (1.1) has an energy integral (Hamiltonian)

$$H = \frac{1}{2}(y, y) + V(x) + \frac{1}{2}\int_0^{+\infty} ((\psi, \psi) + \omega^2(\varphi, \varphi))d\omega - \int_0^{+\infty} (x, q\varphi)d\omega \tag{2.5}$$

where $\psi = \partial_t\varphi$. Let the function $f_1 = f_1(t)$ be the Lebesque-integrable in the interval $[0, T]$ and suppose that a solution of problem (1.7) exists in that interval. Defining $f_1 = f_1(t)$ as zero outside the interval and setting $f = f(t) = 0$, we can evaluate the functions $\varphi_0 = \varphi_0(\omega)$, $\psi_0 = \psi_0(\omega)$ from (1.9) (e.g. by using inverse Fourier transforms). However, substitution of x_0, y_0, $\varphi_0(\omega)$, $\psi_0(\omega)$ into formula (2.5) yields integrals with respect to ω that may be divergent. Nevertheless, the integrals in another expression, which is also constant (more precisely, vanishes identically) by virtue of the Hamilton equations, are convergent

$$\Delta H = \frac{1}{2}(y(t), y(t)) + V_1(x(t)) - \frac{1}{2}(y_0, y_0) - V_1(x_0) + \alpha - \int_0^t (f_1(t_1), \ y(t_1))dt, \ t \in [0, T]$$

$$\alpha(t) = \int_0^t dt_1 \int_0^t (y(t_1), \ K(t_1 - t_2)y(t_2))dt_2$$

where we have used (1.6) and (1.9).

Note that the relation $\Delta H = 0$ may be obtained by evaluating the scalar product of Eq. (1.7) and $y = \partial_t x$ and integrating from 0 to t. The more complicated derivation presented above was intended to demonstrate its meaning as an energy conservation law in the broader Hamiltonian system.

Lemma 2. Let $f_1 = f_1(t)$ be a Lebesgue-integrable vector-valued function in the interval $[0, T]$ and let $x = x(t)$ be a solution of problem (1.4), (1.7) in that interval.
Then the following inequality holds for $0 \leqslant t \leqslant T$

$$\frac{1}{2}(y(t), y(t)) + V_1(x(t)) + \alpha(t) \leqslant A_0 + \tau^2 + A_1^{1/2}\tau$$

$$A_0 = V_1(x_0) + \frac{1}{2}(y_0, y_0), \quad A_1 = A_0 - \inf V_1$$

$$\tau = \int_0^t |f_1(t)|dt_1$$

Proof. It follows from the relation $\Delta H = 0$ that

$$\frac{1}{2}(y(t), y(t)) + V_1(x(t)) + \alpha(t) \leqslant A_0 + Z(t) \tag{2.6}$$

$$Z(t) = \int_0^t |f_1(t_1)||y(t_1)|dt_1$$

By the restrictions imposed on the kernel, $\alpha \geqslant 0$. Hence, and by inequality (2.6), we have the estimate

$$\frac{1}{2}(\partial_\tau Z)^2 \leqslant A_1 + Z$$

Applying Gronwall's Lemma [12], we obtain

$$Z \leqslant \tau^2 + A_1^{1/2}\tau$$

Now, using this inequality, one can estimate the right-hand side of (2.6) and obtain the required assertion.

Having an a priori estimate for the solution, one can now prove a global existence and uniqueness theorem.

Theorem 2. Let $f_1 = f_1(t)$ be a locally Lebesgue-integrable and locally bounded function for $t \geqslant 0$. Then problem (1.5), (1.8) has a unique solution over the half-line $t \geqslant 0$.

Proof. Applying Theorem 1 to the sequence of problems

$$\partial_t z(t) = \Phi(z(t)) + \int_{t_0}^t \Psi(t - t_1)z(t_1)dt_1 + \varkappa_1(t)$$

$$\varkappa_1(t) = \varkappa(t) + \int_0^{t_0} \Psi(t - t_1)z(t_1)dt_1$$

one can achieve local continuation of the solution. It must be shown that the procedure indeed yields a solution at all times.

Suppose the solution has been constructed for an interval $0 \leqslant t \leqslant T$. It follows from Lemma 2 and condition (1.2) that, for some positive number A, the inequality $| z(t) | \leqslant A$ holds throughout this time interval.

Define

$$b = \max_{|x| \leqslant A+1} |\nabla V(x)|, \quad b_1 = Ak + \sup_{0 \leqslant t \leqslant 2T} |f_1(t)|$$

$$b = b_0 + b_1 + (A + 1)(k + 1)$$

It follows from Theorem 1 that for all points of the interval $0 \leqslant t \leqslant T$ one can continue the solution forward over an interval of length $\Delta t = \min(T, b^{-1})$. Hence it follows that the solution may be continued to the entire half-line $t \geqslant 0$.

Besides a priori estimates for the solution of Eq. (1.7), which are related to the energy conservation law for the Lagrangian system (1.1), one can derive a priori integral estimates for the time derivatives. These estimates are needed to analyse dissipative effects in Eq. (1.8).

Lemma 3. Let $f_1 = f_1(t)$ be a piecewise continuous, Lebesgue-integrable and bounded vector-valued function in the half-line $t \geqslant 0$. Then, for the solution $x = x(t)$ of problem (1.4), (1.7), the following integrals are convergent

$$\int_0^\infty |\partial_t x(t)|^2 dt < +\infty, \quad \int_0^\infty |\partial_t^2 x(t)|^2 dt < +\infty \tag{2.7}$$

Proof. We complete the definition of $f_1 = f_1(t)$ by setting $f_1(t) = 0$ for $t < 0$. Fix some positive number T and put

$$v_0 = \partial_t^2 x(0), \quad x_1 = x(T), \quad y_1 = \partial_t x(T), \quad v_1 = \partial_t^2 x(T)$$

We choose some non-decreasing function $\mu = \mu(t)$ of class $C^\infty(\mathbb{R})$ satisfying the additional conditions

$$\mu(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 1 \end{cases}$$

We define an auxiliary function

$$u_T(t) = \begin{cases} (x_0 + y_0 t + v_0 t^2/2)\mu(t+1), & t < 0 \\ x(t), & 0 \leqslant t \leqslant T \\ (x_1 + y_1(t-T) + v_1(t-T)^2/2)\mu(T-t+1), & t > T \end{cases}$$

It is obvious that this function is of class $C^1(\mathbb{R})$ and vanishes outside the interval $[-1, T+1]$. In addition, the derivative of the function $v_T = v_T(t) = \partial_t u_T(t)$ is piecewise differentiable.

The function $u_T = u_T(t)$ satisfies the integro-differential equation

$$\partial_t^2 u_T(t) + \nabla V_1(u_T(t)) + \int_{-1}^{t} K(t-t_1)\partial_t u_T(t_1)dt_1 = f_T(t) + f_1(t) \tag{2.8}$$

An expression for the piecewise continuous function $f_T = f_T(t)$ may be calculated from the expression for $u_T = u_T(t)$ and from Eqs (1.7) and (2.8). In particular, for $0 \leqslant t \leqslant T$ we have

$$f_T(t) = \int_{-1}^{0} K(t-t_1)\partial_t u_T(t_1)dt_1 \tag{2.9}$$

By Lemma 2, the solution $x = x(t)$ of Eq. (1.7) and the time derivative $\partial_t x = \partial_t x(t)$ are bounded for $t \geqslant 0$

$$|x(t)| \leqslant C_0, \quad |\partial_t x(t)| \leqslant C_1 \tag{2.10}$$

Here and below, C_n denotes positive numbers independent of the parameter T. Using (1.7) and (2.10), we obtain the estimate

$$|\partial_t^2 x(t)| \leqslant C_2 \tag{2.11}$$

Relations (2.9)–(2.11) yield the following estimate

$$\int_{-1}^{T+1} |f_T(t)|\,dt < C_3 \tag{2.12}$$

We define a linear operator Γ acting from the space $H^1(\mathbb{R})$ into the space $H^{-1}(\mathbb{R})$ (for the definition of the Hardy spaces $H^n(\mathbb{R})$, see [5])

$$(\Gamma v)(t) = -\partial_t^2 v(t) - \nabla\nabla V_1(u_T(t))v(t) - \int_{-\infty}^{t} K(t-t_1)\partial_t v(t_1)dt_1$$

It follows from (2.8) that the function $v_T = v_T(t)$ satisfies the equation

$$(\Gamma v_T)(t) = -\partial_t(f(t) + f_T(t)) \tag{2.13}$$

Relations (2.10)–(2.12), as well as the fact that $f(t)$ is integrable, imply the inequality

$$\int_{-1}^{T+1} (\partial_t v_T(t), (f(t) + f_T(t)))dt < C_4 \tag{2.14}$$

From (2.13) and (2.14) we obtain

$$C_3 > \int_{-\infty}^{+\infty} v_T(t)(\Gamma v_T)(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\Omega^2 |v_{TF}(\Omega)|^2 - i\Omega(v_{TF}(\Omega), K_F(\Omega)v_{TF}(\Omega))d\Omega -$$

$$- \int_{-\infty}^{+\infty} (v_T(t), \nabla\nabla V_1(u_T(t))v_T(t))dt \tag{2.15}$$

We have the inequalities

$$\|\nabla\nabla V_1(u_T(t))\| < C_5, \quad \|K_F(\Omega)\| < C_6$$

which may be substituted into (2.15), to obtain

$$C_4 > (2\pi)^{-1} \int_{-\infty}^{+\infty} (\Omega^2 - C_5 - C_6 |\Omega|) |v_{TF}(\Omega)|^2 \, d\Omega \tag{2.16}$$

We choose some positive quantity Ω_0 so that for $|\Omega| \geqslant \Omega_0$

$$(\Omega^2 - C_5 - C_6 |\Omega|) > C_7 > 0$$

It then follows from (2.6) that

$$2\pi C_4 > C_7 \int_{|\Omega| > \Omega_0} |v_{TF}(\Omega)|^2 \, d\Omega - (C_5 + C_6 |\Omega_0|) \int_{|\Omega| < \Omega_0} |v_{TF}(\Omega)|^2 \, d\Omega \tag{2.17}$$

We now apply Lemma 2 to Eq. (2.8). As a consequence of this we obtain the inequality

$$\int_{-1}^{T+1} dt_1 \int_{-1}^{T+1} dt_2 (v_T(t_1), \quad K(t_1 - t_2) v_T(t_2)) \leqslant C_8$$

Hence, taking (1.7) into consideration, we obtain the limit

$$\int_{|\Omega| < \Omega_0} |v_{TF}(\Omega)|^2 \, d\Omega < \frac{4\pi}{\delta} C_8, \quad \delta = \min_{|\Omega| \leqslant \Omega_0} (\rho(\Omega)) > 0$$

Combining this limit with (2.17), we obtain

$$\int_{-\infty}^{+\infty} |v_{TF}(\Omega)|^2 \, d\Omega < C_9 \tag{2.18}$$

Returning again to inequality (2.17) and using (2.18), we obtain

$$\int_{-\infty}^{+\infty} \Omega^2 |v_{TF}(\Omega)|^2 \, d\Omega < C_{10} \tag{2.19}$$

It now follows from (2.18) and (2.19) that

$$\int_0^T |\partial_t x(t)|^2 \, dt < \frac{1}{2\pi} C_9$$

$$\int_0^T |\partial_t^2 x(t)|^2 \, dt < \frac{1}{2\pi} C_{10}$$

Since the right-hand sides of these inequalities are independent of T, this proves that the integrals (2.7) are convergent.

3. THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS FOR LONG TIMES

We will show that Eq. (1.7) describes effects typical of dissipative systems, such as damped oscillations These effects are valid regardless of the previously proved equivalence of processes (1.17) to certain processes in an infinite-dimensional Hamiltonian system when there are no external forces. Thus, the results of this section constitute a counter-example to the problem of recurrence in infinite-dimensional Hamiltonian mechanics.

Theorem 3. Let $f_1 = f_1(t)$ be a piecewise continuous function, bounded and Lebesgue-integrable for $t \geqslant 0$. Assume that the potential $V_1 = V_1(x)$ has a finite set of critical points. Then the solution $x = x(t)$ of problem (1.4), (1.7) converges to one of the critical points of the potential.

Proof. By Lemma 2, the solution $x = x(t)$ and its derivative are bounded at all times

$$|x(t)| \leqslant A_0, \quad |\partial_t x(t)| \leqslant A_1$$

We rewrite Eq. (1.7) in the equivalent form

$$\partial_t^2 x(t) = -\nabla V_1(x(t)) + \varkappa_*(t) \tag{3.1}$$
$$\varkappa_*(t) = \varkappa_1(t) + \varkappa_2(t)$$
$$\varkappa_1(t) = -\int_0^t K(t-t_0)\partial_t x(t_0)dt_0$$
$$\varkappa_2(t) = f_1(t)$$

The function $\kappa_2(t)$ is square-integrable, since it is integrable and bounded. The function $\kappa_1(t)$ is also square-integrable, as a convolution of functions with the analogous property.

The proof will be carried out by *reductio ad absurdum*.

Suppose a positive constant C_0 exists such that there is an infinite sequence of times t_k at which

(a) $t_{k+1} > t_k + 1$

(b) $|\nabla V_1(x_k)| \geqslant C_0, x_k = x(t_k)$.

We choose a sufficiently small positive number $\varepsilon < 1$ so that

$$C_0 - A_2 A_1 \varepsilon \geqslant C_1 > 0, \quad A_2 = \max_{|x| \leqslant A_0} \| \nabla\nabla V_1(x) \|$$

Then $|\nabla V_1(x(t))| \geqslant C_1$ for $t_k \leqslant t \leqslant t_k + \varepsilon$ Hence, by Eq. (3.1), we obtain the limit

$$\int_{t_k}^{t_k+\varepsilon} |\partial_t^2 x(t)|^2 \, dt \geqslant \frac{1}{2} C_1^2 \varepsilon - \int_{t_k}^{t_k+\varepsilon} |\varkappa_*(t)|^2 \, dt$$

Since the integral

$$\int_0^\infty |\varkappa_*(t)|^2 \, dt$$

is convergent, so is the integral

$$\int_0^\infty |\partial_t^2 x(t)|^2 \, dt$$

This in turn contradicts the results of Lemma 3.

If the conditions of Theorem 3 are satisfied, the solution converges to some critical point x_* of the potential $V_1(x)$. Suppose this critical point is non-degenerate. Applying, if necessary, a translation along the vector x_*, we may assume without loss of generality that $x_* = 0$. At sufficiently large time values, one can linearize problem (1.4), (1.7) (if necessary applying a time shift). The result is a linear equation

$$\partial_t^2 x(t) + Lx(t) + \int_0^t K(t-t_1)\partial_t x(t_1)dt_1 = f_1(t) \tag{3.2}$$

where $L = \nabla\nabla V_1(0)$. This equation may be solved by the method of Fourier–Laplace transforms. Changing to Fourier transforms and using condition (1.4), we reduce Eq. (3.2) to the form

$$T(\Omega)x_F(\Omega) = f_{1F}(\Omega) + y_0 + i\Omega x_0$$
$$T(\Omega) = -\Omega^2 + L + i\Omega K_F(\Omega)$$

The matrix-valued function $T(\Omega)$ is non-singular on the entire real axis. Indeed, $T(0) = L$. If $\Omega \neq 0$, it follows from inequality (1.17) that

$$i\Omega^{-1}T(\Omega)^+ - i\Omega^{-1}T(\Omega) \geqslant \rho(\Omega)\,\mathrm{id}_{\mathbf{R}^N}$$

Hence, one can write the solution for $t \geqslant 0$ as a Fourier integral, understood in the sense of the principal value

$$x(t) = \frac{1}{2\pi}\int \exp(i\Omega t)x_F(\Omega)d\Omega \tag{3.3}$$
$$x_F(\Omega) = T^{-1}(\Omega)(f_{1F}(\Omega) + y_0 + i\Omega x_0)$$

The integration in (3.3) is performed along the real axis, since the contribution from the poles in the lower complex half-plane vanishes due to our assumptions.

Generally speaking, formula (3.3) says little about the rate at which the solution tends to zero. However, subject to certain additional assumptions, one can derive fairly strong estimates. Thus, suppose that for some natural number $n \geqslant 1$ the integrals

$$\int_0^\infty t^n \| K(t) \| dt < +\infty, \quad \int_0^\infty t^n | f_1(t) | dt < +\infty$$

are convergent. Then one can integrate by parts in (3.3) for $t > 0$

$$x(t) = \frac{1}{2\pi}\left(\frac{i}{t}\right)^n \int \exp(i\Omega t) \frac{d^n}{d\Omega^n} x_F(\Omega) d\Omega$$

to obtain the asymptotic estimate

$$| x(t) | = O(t^{-n})$$

4. CONCLUSION

Thus, systems of the type of a multi-dimensional non-linear oscillator with relaxation fall into the category of infinite-dimensional Hamiltonian systems. At the same time, these systems exhibit dissipative effects typical of ordinary mechanical systems with friction. It seems that this result may serve as a bridge between Hamiltonian mechanics and the ordinary mechanics of dissipative systems, since all time-local dissipative systems in the real world are limiting cases of systems with relaxation.

The assumption adopted in this paper that a system with heredity has only a finite number of degrees of freedom is not essential. The results as a whole extend to infinite-dimensional systems with heredity (e.g. visco-elastic continuous media), but such cases require more complicated mathematical treatment.

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